INVERSE DYNAMICS OF 2-DOF PLANAR PARALLEL MANIPULATORS WITH PRISMATIC ACTUATORS

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ABSTRACT

This paper presents a recursive formulation for the inverse dynamics of a generic two-Degree of Freedom planar parallel manipulator with prismatic actuators, which find applications in positioning, contouring, etc. with high accuracy and precision. The method is based on the Newton-Euler equations of motion and the Decoupled Natural Orthogonal Complement matrices, which relate the angular and linear velocities of the rigid bodies of the system to the associated joint rates. The dynamic modeling developed is used to study the variation of actuating forces of two such manipulators. The results are useful for the selection of a suitable type and to decide the proper actuator ratings.

Keywords- Inverse dynamics, DeNOC, Parallel manipulators, Prismatic actuators

INTRODUCTION

The dynamic analysis of parallel kinematic machines (PKMs) has drawn much attention in recent years as these machines meet the requirement of high dynamic performance. A generic 2-Degree of Freedom (DOF) planar parallel manipulator with prismatic actuators is shown in Fig. 1. This is one of the simplest and most common versions of PKMs that find applications in positioning, contouring etc. with high accuracy and precision. It consists of two distinct rails. The sliders move along their rails, whereas the legs of constant length are connected to the sliders through revolute joints. Other ends of both the legs are connected together through another revolute joint. Actuation of the sliders drives the end-effector attached to one of the legs at a suitable position. Most of the researchers studied, extensively, kinematics of PKMs while fewer papers can be found on the dynamics [1]. Formulation methods to derive the dynamics equations of motion fall into two main categories, namely, (i) Euler-Lagrange, and (ii) Newton-Euler [2].

Unlike in serial manipulators, in multi-loop mechanisms, the joint variables are not independent, as there are non-linear loop closure constraints that need to be satisfied. The closed-loops of the system make the dynamic computations more expensive. In order to take the advantages of the efficient algorithms of the serial chain manipulators, the most
common method for dealing with the closed-loop kinematics is to cut the loop, introduce Lagrange multipliers to substitute for the cut joints so that the efficient recursive scheme [3] for the open-chain system can be utilized. The Natural Orthogonal Complement (NOC) introduced in [3] belongs to the class of projection methods for dynamic modeling, where the NOC of a serial chain is written as a product of two matrices, one is block diagonal and the other one is lower block diagonal. So, the term Decoupled Natural Orthogonal Complement (DeNOC) is used. Although recursive kinematics algorithms for serial chains have had a long history, a recursive algorithm for the forward kinematics of closed-chain systems, particularly, in minimal Ordinary Differential Equation (ODE) form first appeared in [4]. In this work, Saha and Schiehlen [4] showed that the NOC of a parallel manipulator may be split into three matrices, namely, the lower-block triangular, the full-block, and the block-diagonal. Later, in [5], it is shown that the methodology can be suitably adopted for parallel computations in multi-platform computers. In this paper, the application of the DeNOC to the closed loop-chain with prismatic actuators is proposed. The methodology is illustrated with a 2-DOF planar parallel manipulator.

**BASIC CONCEPTS OF DeNOC FORMULATION**

Figure 2 shows two rigid links connected by a kinematic pair, revolute or prismatic. The notations are according to [3]. The mass center of the $i$th link is at $C_i$ while that of link $i-1$ is at $C_{i-1}$. The axis of the $i$th pair is represented by the unit vector $e_i$. A frame $F_i$ with origin $O_i$ and axes $x_i$, $y_i$, and $z_i$, is attached to link $i$, such that $z_i$ is along $e_i$. The global inertial reference frame $F_0$ with axes $X$, $Y$, and $Z$ is attached to the base of the manipulator. All quantities, unless otherwise specified, are represented in this global absolute frame. Further, the position vectors $d_i$ from $O_i$ to $C_i$, and $r_{i-1}$ from $C_{i-1}$ to $O_i$ are the 3-dimensional Cartesian vectors. The 6-dimensional vectors of twist and wrench of link $i$ at its mass center $C_i$ are defined as

$$ t_i = [\omega_i^T \ v_i^T]^T; \quad w_i = [n_i^T \ f_i^T]^T \quad (1) $$

![Fig. 1 A generic 2-DOF planar parallel manipulator with prismatic actuators](image1)

![Fig. 2 Two bodies connected by a lower kinematic pair](image2)
where \( \omega_i, v_i, n_i \), and \( f_i \) are the three-dimensional angular velocity, linear velocity of \( C_i \), moment about \( C_i \) and force vectors, respectively, associated with link \( i \).

The Newton-Euler equations of motion for link \( i \) can be written as [2]

\[
\mathbf{w}_i = \mathbf{M}_i \mathbf{t}_i + \mathbf{W}_i \mathbf{t}_i
\]

where \( \mathbf{M}_i \equiv \text{diag.} \begin{bmatrix} 1, m_i \end{bmatrix} \) and \( \mathbf{W}_i \equiv \text{diag.} \begin{bmatrix} \Omega_i, \mathbf{O} \end{bmatrix} \) are \( 6 \times 6 \) matrices and \( \mathbf{t}_i \) is the 6-dimensional twist rate vector, and \( \Omega_i : 3 \times 3 \) inertia tensor about \( C_i \) and \( m_i : \text{mass of link } i \).

For a multi-body system with \( n \) rigid links coupled by kinematic pairs, one may write

\[
\mathbf{t} = [\mathbf{t}_1^T, \ldots, \mathbf{t}_n^T]^T; \quad \mathbf{w} = [\mathbf{w}_1^T, \ldots, \mathbf{w}_n^T]^T
\]

The set of Newton-Euler equations for the entire unconstrained system may be written as

\[
\mathbf{w} = \mathbf{M} \mathbf{t} + \mathbf{WMt}
\]

At this juncture, DeNOC matrices are obtained as the transformation from the joint rates to the twist of all the bodies. Note that the twist of the mass center of link \( i \) at \( C_i \) can be written recursively in terms of the twist of the mass center of link \( i-1 \), i.e., \( C_{i-1} \) as

\[
\mathbf{t}_i = \mathbf{A}_{i,i-1} \mathbf{t}_{i-1} + \mathbf{p}_i \dot{\theta}_i
\]

where the 6x6 twist propagation matrix, \( \mathbf{A}_{i,i-1} \) and the 6-dimensional twist generator vector, \( \mathbf{p}_i \), are given by

\[
\mathbf{A}_{i,i-1} = \begin{bmatrix} 1 & \mathbf{O} \\ \mathbf{A}_i & 1 \end{bmatrix}; \quad \mathbf{p}_i = \begin{bmatrix} \mathbf{e}_i \\ \mathbf{D}_e \mathbf{e}_j \end{bmatrix} \quad \text{for revolute joint}; \quad \mathbf{p}_j = \begin{bmatrix} 0 \\ \mathbf{e}_j \end{bmatrix} \quad \text{for prismatic joint}
\]

\( \Lambda_i \) being the cross product matrix of \(- (\mathbf{r}_{i-1} + \mathbf{d}_i)\). The twist \( \mathbf{t}_i \) is thus the sum of twist \( \mathbf{t}_{i-1} \) and the twist generated at the joint, \( i \). The derivation of the DeNOC matrices and their use to reduce the uncoupled Newton-Euler dynamic equations of motion, Eq. (4), to a reduced independent set for the coupled system is detailed in the following sections.

**DYNAMIC MODELING USING DeNOC**

The generic 2-DOF planar parallel manipulator, shown in Fig. 1, may be discretized into two serial chains such that the end-effector of each chain lies at \( P \) as shown in Fig. 3. The points, \( C_1^2 \) and \( C_2^2 \) are the mass centers of the 2nd leg (or link) belonging to the 1st and 2nd chains, respectively. For the position analysis, consider the closed loop \( O-A_1-P-A_2-O \),

\[
\mathbf{O} \mathbf{A}_1^+ \mathbf{A}_1 \mathbf{P} = \mathbf{O} \mathbf{A}_2^+ \mathbf{A}_2 \mathbf{P}
\]
Equation (7) offers closed form solution to both forward and inverse position analysis.

For the velocity analysis as the manipulator is planar, the twist, \( \mathbf{t} \equiv [\omega \; \mathbf{v}^T]^T \), and wrench \( \mathbf{w} \equiv [n \; \mathbf{f}^T]^T \), would be 3-dimensional vectors where \( \omega \) is the angular velocity, \( \mathbf{v} \) is the 2-dimensional linear velocity vector, \( n \) is the angular moment and \( \mathbf{f} \) is the 2-dimensional force vector. The twist of the end-effector of each chain can be written as [3]

\[
\mathbf{t}_p = \mathbf{A}_{p2} \mathbf{t}_2
\]  

(8)

where \( \mathbf{A}_{p2} = \begin{bmatrix} 1 & 0^T \\ \mathbf{E}r_2 & 1 \end{bmatrix} \); \( \mathbf{E} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \); and \( \mathbf{t}_2 \) is the twist of second link at its mass center, which, for each serial chain, can be expressed in terms of its preceding link as

\[
\mathbf{t}_2 = \mathbf{A}_{21} \mathbf{t}_1 + \mathbf{p}_2 \dot{\theta}_2
\]  

(9)

where \( \mathbf{A}_{21} = \mathbf{E} \mathbf{r}_1 + \mathbf{d}_2 \); and \( \mathbf{p}_2 = \begin{bmatrix} 0 \\ \mathbf{E} \mathbf{d}_2 \end{bmatrix} \). The 3x3 matrix, \( \mathbf{A}_{21} \), is the twist propagation matrix, and \( \mathbf{p}_2 \) is the twist generator, as introduced in Eq. (6), Moreover, \( \mathbf{t}_1 \) is the twist of link-1 with respect to its mass centre; \( \dot{\theta}_2 \) is the relative angular joint of velocity of link-2. Substituting \( \mathbf{t}_2 \) into Eq. (8), we obtain

\[
\mathbf{t}_p = \mathbf{A}_{p2} (\mathbf{A}_{21} \mathbf{t}_1 + \mathbf{p}_2 \dot{\theta}_2)
\]  

(10)

Equation (10) may be solved to determine the unactuated joint rate \( \dot{\theta}_2 \) as

\[
\begin{align*}
\theta_2^1 &= \theta_2^1 - \theta_2^2 \\
d_2^1 &= 0 \\
d_1^2 &= 0 \\
r_1^2 &= 0 \\
r_1^1 &= 0 \\
\end{align*}
\]

superscripts refer to the chain to which they belong

Fig. 3 Discretized version of the manipulator
where \( \hat{\theta}_2 = \frac{\hat{p}_2^T}{\hat{\theta}_2} (\mathbf{t}_p - A_p \mathbf{t}_1) \) \hspace{1cm} (11)

Substituting \( \hat{\theta}_2 \) and \( \mathbf{t}_1 \equiv \mathbf{p}_1 \hat{b}_1 \) into Eq. (10) we get

\[
\Phi_2 \mathbf{t}_p = \Phi_2 A_p \mathbf{p}_1 \hat{b}_1
\]

(12)

in which \( \Phi_2 = 1 - \tilde{\mathbf{p}}_2 \tilde{\mathbf{p}}_2^T / \hat{\theta}_2 \).

Writing Eq. (12) for each open chain and adding, we get:

\[
K \mathbf{t}_p = \Phi A P \hat{\theta}_{ac}
\]

(13)

Where \( K \equiv \Phi_1^2 + \Phi_2^2 \): 3x3 matrix; \( \Phi \equiv [\Phi_1^2 \quad \Phi_2^2] \): 3x6 matrix; \( A \equiv \text{diag.} \ (A_{p1}^1, A_{p1}^2) \): 6x6 matrix; \( P \equiv \text{diag.} \ (\mathbf{p}_1^1, \mathbf{p}_1^2) \): 6x2 matrix; and \( \hat{\theta}_{ac} = [\hat{\theta}_1^1 \quad \hat{\theta}_2^2]^T \): 2-dimensional vector.

Equation (11) may, now, be re-written to find the unactuated joint rates as

\[
\hat{\theta}_2 = \frac{\hat{p}_2^T}{\hat{\theta}_2} (K^{-1} \Phi A P \hat{\theta}_{ac} - A_{p1} \mathbf{p}_1 \hat{b}_1)
\]

(14)

We note that Eq. (14) is general and applicable to each open chain. Applying this equation to both the chains, and arranging in matrix form yields the relationship between the joint rates and the actuated joint rates as

\[
\hat{\theta} = \mathbf{PLA} \hat{\theta}_{ac}
\]

(15)

where joint rate vector, \( \hat{\theta} = \begin{bmatrix} \hat{\theta}_1^T & \hat{\theta}_2^T \end{bmatrix}^T \) in which \( \hat{\theta}_1 = [\hat{b}_1^1 \quad \hat{\theta}_2^1]^T \) and \( \hat{\theta}_2 = [\hat{b}_2^1 \quad \hat{\theta}_2^2]^T \);

\[
\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & 0 \\ 0 & \tilde{\mathbf{P}} \end{bmatrix}_{4x12}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 \\ \mathbf{S}_1 - 1 & \mathbf{S}_2 \\ 0 & 1 \\ \mathbf{S}_1 & \mathbf{S}_2 - 1 \end{bmatrix}_{12x6}, \quad \mathbf{A} = \begin{bmatrix} \tilde{\mathbf{p}}_1^1 & 0 \\ 0 & \tilde{\mathbf{p}}_2^2 \end{bmatrix}_{6x2} ; \quad \text{and} \quad \hat{\mathbf{p}}_j = A_{p_j} \mathbf{p}_j^j \]

\( \mathbf{P}_j = \text{diag.} \ [\tilde{\mathbf{p}}_j^T / \delta_j^1, \tilde{\mathbf{p}}_j^T / \delta_j^2], \delta_j^1 = \tilde{\mathbf{p}}_j^T \tilde{\mathbf{p}}_j^j ; \text{and} \ \mathbf{S}_j = K^{-1} \Phi_j^j \) for \( i = 1, 2, j = 1, 2. \)

Differentiating Eq. (10) and, adopting a process similar to the one discussed for velocity analysis, we get

\[
\Phi_2 \mathbf{t}_p = \Phi_2 a_1 + \Phi_2 a_2
\]

(16)
where $a_1 \equiv [\hat{A}_{p2} t_2 + A_{p2} (\hat{A}_{21} t_1 + A_{21} \hat{p}_1 \hat{b}_1 + \hat{p}_2 \hat{b}_2)]$ and $a_2 \equiv A_{p1} p_1 \hat{b}_1$. Adding Eq. (16) for each open chain we get

$$K \dot{p}_r = \Phi [a_1^T a_1^2] + \Phi \Lambda P \dot{\theta}_{ac}$$

(17)

where $\dot{\theta}_{ac} \equiv [\dot{b}_1^T \dot{b}_2^T]^T$. Equation (17) can be used for the acceleration analysis.

The Newton-Euler equations, Eq. (4), for each open chain may be expressed as

$$M \ddot{t} + \dot{t} = w^{ac} + w^w + w^g + w^c$$

(18)

where $M$ is the 6x6 mass matrix, $t$ is the 6-dimensional twist vector of the whole chain, $w^{ac}$ is the wrench applied by the actuators, $w^w$ is the wrench applied at the end effector, $w^g$ is the gravity wrench, and $w^c$ are the constraint wrenches. All the wrenches are 6-dimensional vectors. The twist vector may be written as [3]

$$t = N_l N_d \dot{\theta}$$

(19)

where $N_l$ and $N_d$ are the DeNOC matrices, and $\dot{\theta}$ is the joint rate vector, not recursively independent, of each chain. For our manipulator, for each open chain,

$$N_l = \begin{bmatrix} 1 & 0 \\ A_{21} & 1 \end{bmatrix}; \quad N_d = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}; \quad \text{and} \quad \dot{\theta} = [\dot{b}_1^T \dot{b}_2^T]^T$$

(20)

As the constraint wrenches do not develop any power, $t^T w^c = 0$. Pre-multiplying both sides of Eq. (18) by $N_d^T N_l^T$, and noting that $M = 0$ (for planar manipulators), we get

$$N_d^T N_l^T \ddot{t} + \dot{t} = \ddot{\tau} + N_d^T N_l^T (w^w + w^g)$$

(21)

where $\ddot{\tau} \equiv N_d^T N_l^T w^{ac}$ is the joint torque / force vector for each open chain and is given by $\ddot{\tau} = [\ddot{\tau}_1 \ddot{\tau}_2]^T$. Differentiating Eq. (21) and substituting the expression for $t$ in Eq. (21), we get, for each chain,

$$I \ddot{\theta} + C \dot{\theta} = \ddot{\tau} + \tau^G$$

(22)

where $I \equiv N_d^T N_l^T M N_l N_d$, 2x2 generalized inertia matrix of the chain,

$$C \equiv N_d^T N_l^T (M N_l N_d + M N_d N_l)$$

(23)

and $\tau^G \equiv N_d^T N_l^T (w^w + w^g)$. Considering the dynamics equation for each chain, i.e., Eq. (22), and coupling them with Lagrange multipliers [5], and projecting the joint torques on to minimal-coordinate space one can obtain the actuated forces for the manipulator under study as

$$\tau_{ac} = [\tau_{ac}]^T [(S^l)^T (S^l)^T] - \hat{p}_1^T - \hat{p}_2^T]$$

(23)
where $\dot{\mathbf{p}}_2^{j} = \ddot{x}_2^{j} \dot{p}_2^{j} / \delta_2^{j}$, for $j = 1, 2$. The recursive calculations for each chain which yield the force $\tau_1^{j}$ in Eq. (23) are as follows:

$$\begin{align*}
\mathbf{y}_2^{j} & = \mathbf{M}_2^{j} \mathbf{i}_2^{j} - \mathbf{w}_2^{G,j} ; \\
\ddot{x}_2^{j} & = \left( \mathbf{M}_1^{j} \mathbf{I}_{1}^{j} - \mathbf{w}_1^{G,j} \right) + \mathbf{A}_{21}^{j T} \mathbf{y}_2^{j} ; \\
\ddot{x}_1^{j} & = \mathbf{p}_1^{j T} \mathbf{y}_1^{j} 
\end{align*}$$

(24)

where $\mathbf{w}_i^{G,j} = \mathbf{w}^{w,j} + \mathbf{w}^{g,j}$, for $i = 1, 2$ (joint), $j = 1, 2$ (chain).

**CASE STUDIES**

The dynamic modeling developed has been implemented for the two typical cases of a 2-DOF planar parallel manipulator with prismatic actuators, having different rail-arrangements, as shown in Figures 4 and 5. The legs and actuators are considered to be identical in both the cases. Lengths of the rails are considered such that the size of the manipulator [6], which is the smallest sized rectangle that contains the total workspace of the manipulator and the rails, is same in both the cases. The specifications of the manipulators are: Mass of each actuator: 50.0 kg; Mass of each leg: 200.0 kg; Length of each leg: 2.0 m; Lengths of rails in the manipulator with parallel rails: 1.5 m. Lengths of rails in the manipulator with collinear rails are found to be 1.615 m.

The inverse dynamics is performed using MATLAB when the end-effector starts at point $P_0$ (Figures 4 and 5) and traces a circular path in counter clockwise direction with a constant speed, $V = 0.25$ m/s. The centre of the circle is considered to be at a distance of $h/3$ from the bottom of the workspace, $h$ being the workspace range along Y-axis. Radius of the circle, $R = 0.2$m. The gravity, $g$ is assumed to act in the –Y direction. The variation of the actuating forces with the angle $\beta$ (the position of end-effector) during contouring, for both the manipulators, is shown in Figures 6 and 7. Note that the maximum actuation forces required in case of the manipulator with collinear rails are almost $1/3^{rd}$ of those required in case of the manipulator with parallel rails. This may be due to the fact that in the former, actuator masses moves horizontally and, then, do not affect the dynamics.

**CONCLUSIONS**

In this work, a method based on the Decoupled Natural Orthogonal Complement is introduced for the inverse dynamics analysis of 2-DOF planar parallel manipulators with prismatic actuators. The set of sub-modal dynamics is projected onto the minimal-coordinate space, i.e., the space of feasible motions. The decoupled form of the natural orthogonal complement is used here to exploit the recursive nature of the inverse dynamics algorithm that results in efficient computations. The dynamic modeling, presented in this work, has been implemented for two typical cases of the 2-DOF planar parallel manipulator representing different rail-arrangements. The results depict the
strong dependence of actuating forces on the rail-arrangement. Moreover, these results are useful for the estimation of power ratings of actuator motors.

Fig. 4 Manipulator with parallel rails and its workspace

Fig. 5 Manipulator with collinear rails and its workspace

Fig. 6 Variation of actuator forces for the manipulator with parallel rails

Fig. 7 Variation of actuator forces for the manipulator with collinear rails

REFERENCES