Modified SSTD method to account for harmonic excitations during Operational Modal Analysis

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Abstract

In-operation modal testing is a procedure, which allows to extract modal parameters of a structure in operational condition. It is based on the assumption that input to the structure is stationary white noise. In practice however, many structure's input force can not be assimilated to white noise input. In some of the cases, input force can be compared to the combination of white noise and harmonic excitations. Harmonic excitations can occur due to components like unbalanced rotors or fluctuating forces in electric actuators. The usual way to compute modal parameters in the presence of harmonic excitations is to treat harmonically excited frequencies to be virtual eigenfrequencies of the structure. However, if the frequency of the harmonic input is close to an eigenfrequency of the system, operational modal analysis procedures fail to identify the modal parameters properly. Correlation functions of the response signals can be treated as a impulse response functions as stated in the Natural Excitation Technique (NExT). Time-domain method like Single Station Time Domain (SSTD) can be used to process the correlation functions to compute modal parameters. In this paper a modified SSTD method is proposed, which can be applied to include the effect of purely harmonic vibrations, assuming the harmonic frequencies are known *a priori*. We illustrate the efficiency of the proposed approach on the experimental example of a plate structure excited by multi-harmonic loads superposed on random excitation.

1 Introduction

In-operation modal analysis modal parameters of structures can be measured without knowing the input excitation to the system. It is therefore a very valuable tool to analyze structures submitted to excitation generated by their own operation. Presently, operational modal analysis procedures are limited to the case when excitation to the system is stationary white-noise. There are different ways to identify modal parameters in that case. One approach called the Natural Excitation Technique (NExT) [1] consists in computing correlations between the response signals and observe that they can be compared to impulse responses of the system. Hence, the output correlation functions can be processed as the impulse responses function of the system in order to extract modal parameters. Standard time-domain identification techniques such as the Least Square Complex Exponential method(LSCE) [10], the Eigenvalue Realization Algorithm(ERA) [11] and the Ibrahim Time Domain method [7–9] can be applied as identification techniques.

In many applications such as wind turbines, cars and ships harmonic excitations are present in addition to random loads due to unbalance masses in rotating components or due to aerodynamic and electrical forces. A straightforward way to deal with harmonic excitations consists in considering the harmonic response as a virtual non-damped eigenmode of the system. So while doing modal identification, one should spot those virtual, non-damped, modes and identify them as arising from harmonic excitations. In practice however if the excitation frequencies are close to natural frequencies of the system, identified natural frequencies and associated damping can no longer be measured accurately by the algorithms. Indeed, in that case, the harmonic response masks the actual eigen-response and classical identification methods are not well suited to separate between harmonic and eigen-component.

In this paper, we will be developing a method based on the SSTD algorithm [6], itself a variant of the Ibrahim Time Domain (ITD) approach. We will assume that the harmonic frequencies are known *a priori*. Note that harmonic excitation frequencies can be found easily in practice given the nature of the machine or using pre-processing of the measured data. Experimental results obtained with the modified SSTD will be compared to results obtained with standard SSTD method in the presence of harmonics.

In earlier papers [2–5], few methods to take into account harmonic excitations in OMA were proposed. In the present paper, we use similar ideas but now based on the Single Station Time Domain (SSTD) algorithm [6], itself a variant of the Ibrahim Time Domain (ITD) method.

2 Single Station Time Domain method

The SSTD method is a variant of the ITD method. In the original ITD method, at least $2N$ response locations need to be measured to identify a modal of order N. In the SSTD, identification can be carried out based on one single response. In terms of complex modes, correlation functions between two measured responses to random excitation can be written as [1]:

$$x(k\Delta t) = x_k = \sum_{r=1}^{2N} a_r e^{s_r k\Delta t}$$

the x_k is the discrete correlation function at k^{th} discrete time, a_r is a constant associated with r^{th} mode and s_r is the r^{th} complex frequency. Now x_k can again be written for different shifted starting time samples as:

$$\underset{\substack{(\mathrm{2N \times L}) \\ [\boldsymbol{X}]}}{\begin{bmatrix} x_1 & \dots & x_{\mathrm{L}} \\ x_2 & \dots & x_{\mathrm{L+1}} \\ \vdots & \dots & \vdots \\ x_{\mathrm{2N}} & \dots & x_{\mathrm{L+2N-1}} \end{bmatrix}} = \underset{\substack{(\mathrm{2N \times 2N}) \\ [\boldsymbol{A}]}}{\begin{bmatrix} a_1 & \dots & a_{\mathrm{2N}} \\ a_1 e^{s_1 \triangle t} & \dots & a_{\mathrm{2N}} e^{s_{\mathrm{2N}} \triangle t} \\ \vdots & \vdots & \vdots \\ a_1 e^{s_1 (\mathrm{2N-1}) \triangle t} & \dots & a_{\mathrm{2N}} e^{s_{\mathrm{2N}} (\mathrm{2N-1}) \triangle t} \end{bmatrix}} \underset{\substack{(\mathrm{2N \times L}) \\ [\boldsymbol{\Lambda}]}}{\begin{bmatrix} e^{s_1 t_1} & \dots & e^{s_1 t_{\mathrm{L}}} \\ e^{s_2 t_1} & \dots & e^{s_2 t_{\mathrm{L}}} \\ \vdots & \dots & \vdots \\ e^{s_{\mathrm{2N}} t_1} & \dots & e^{s_{\mathrm{2N}} t_{\mathrm{L}}} \end{bmatrix}} \tag{1}$$

where $t_k = k\Delta t$, N is the total number of modes considered for the identification and L is the number of correlation values per row. The modal order N is usually not known *a priori* but it is a general practice to compute poles at different modal orders and at last correct poles and number of modes are chosen from convergence of poles in the stability diagram.

We can write a similar equation by shifting all the discrete response values by Δt as follows:

$$
\underbrace{\begin{bmatrix} x_2 & ... & x_{L+1} \\ x_3 & ... & x_{L+2} \\ \vdots & ... & \vdots \\ x_{2N+1} & ... & x_{L+2N} \end{bmatrix}}_{[\hat{\boldsymbol{X}}]} = \underbrace{\begin{bmatrix} a_1 e^{s_1 \triangle t} & ... & a_{2N} e^{s_{2N} \triangle t} \\ a_1 e^{s_1 2\triangle t} & ... & a_{2N} e^{s_{2N} 2\triangle t} \\ ... & \vdots\vdots\vdots & \vdots \\ a_1 e^{s_1 2N\triangle t} & ... & a_{2N} e^{s_{2N} 2N\triangle t} \end{bmatrix}}_{[\hat{\boldsymbol{A}}]} \underbrace{\begin{bmatrix} e^{s_1 t_1} & ... & e^{s_1 t_L} \\ e^{s_2 t_1} & ... & e^{s_2 t_L} \\ \vdots & ... & \vdots \\ e^{s_{2N} t_1} & ... & e^{s_{2N} t_L} \end{bmatrix}}_{[\boldsymbol{\Lambda}]} \quad (2)
$$

Now let us define a system matrix $[\boldsymbol{S}]$ such that

$$
[\boldsymbol{S}]\,[\boldsymbol{A}] = \left[\hat{\boldsymbol{A}}\right] \quad (3)
$$

Pre-multiplying equation (1) by $[\boldsymbol{S}]$ and taking account of (3),

$$
[\boldsymbol{S}]\,[\boldsymbol{X}] = [\boldsymbol{S}]\,[\boldsymbol{A}]\,[\boldsymbol{\Lambda}] = \left[\hat{\boldsymbol{A}}\right][\boldsymbol{\Lambda}] \quad (4)
$$

Taking account of (2) one obtains

$$
\underset{(2N \times 2N)}{[\boldsymbol{S}]}\ \underset{(2N \times L)}{[\boldsymbol{X}]} = \underset{(2N \times L)}{\left[\hat{\boldsymbol{X}}\right]} \quad (5)
$$

The system matrix $[\boldsymbol{S}]$ can thus be computed as a least square solution of equation (5). The eigenvalues of the system are then obtained by observing that (3) can be decomposed as:

$$
\underset{(2N \times 2N)}{[\boldsymbol{S}]} \underset{(2N \times 1)}{\left\{ \begin{matrix} a_r e^{s_r 0\triangle t} \\ \vdots \\ a_r e^{s_r (2N-1)\triangle t} \end{matrix} \right\}} = \underset{(2N \times 1)}{\left\{ \begin{matrix} a_r e^{s_r 0\triangle t} \\ \vdots \\ a_r e^{s_r (2N-1)\triangle t} \end{matrix} \right\}} e^{s_r \triangle t} \qquad r = 1, 2....2N \quad (6)
$$

Equation (6) indicates that computing the eigenvalues of $[\boldsymbol{S}]$ yields the roots $e^{s_r \triangle t}$ and thus s_r. Modal frequencies and damping can be computed from the values of s_r.

3 Modified SSTD method to Include Harmonic Excitations

In the presence of harmonic excitations, the induced vibration can be considered as a non-damped mode of the system caused by the white-noise equivalent of force input to the system. Let us assume that the harmonic excitation has a known frequency ω_1. The modal superposition describing the impulse response now includes a forced harmonic part equivalent to a virtual modal response with frequency $s = \pm i\omega_1$ so that $e^{s\Delta t} = e^{\pm i\omega_1 \Delta t} = \cos(\omega_1 \Delta t) \pm i \sin(\omega_1 \Delta t)$. This root being known *a priori* we will force the system matrix $[\boldsymbol{S}]$ to have $e^{\pm i\omega_1 \Delta t}$ as an eigensolution: writing (5) for $s = \pm i\omega_1$ and rearranging, $[\boldsymbol{S}]$ must satisfy

$$
\underset{(2N \times 2N)}{[\boldsymbol{S}]} \underset{(2N \times 2)}{\begin{bmatrix} 0 & 1 \\ \sin(\omega_1 \triangle t) & \cos(\omega_1 \triangle t) \\ : & : \\ \sin((2N\text{-}2)\,\omega_1 \triangle t) & \cos((2N\text{-}2)\,\omega_1 \triangle t) \\ \sin((2N\text{-}1)\,\omega_1 \triangle t) & \cos((2N\text{-}1)\,\omega_1 \triangle t) \end{bmatrix}} = \underset{(2N \times 2)}{\begin{bmatrix} \sin(\omega_1 \triangle t) & \cos(\omega_1 \triangle t) \\ \sin(2\omega_1 \triangle t) & \cos(2\omega_1 \triangle t) \\ : & : \\ \sin((2N\text{-}1)\,\omega_1 \triangle t) & \sin((2N\text{-}1)\,\omega_1 \triangle t) \\ \sin(2N\omega_1 \triangle t) & \cos(2N\omega_1 \triangle t) \end{bmatrix}} \quad (7)
$$

or symbolically

$$
\underset{(2N \times 2N)}{[\boldsymbol{S}]}\ \underset{(2N \times 2)}{\left[\boldsymbol{H}^1\right]} = \underset{(2N \times 2)}{\left[\hat{\boldsymbol{H}}^1\right]} \quad (8)
$$

If m harmonic excitation frequencies exist, (8) can be written generalized to

$$[\boldsymbol{S}]\left[[\boldsymbol{H}^1][\boldsymbol{H}^2]\,..\,[\boldsymbol{H}^m]\right] = \left[\left[\hat{\boldsymbol{H}}^1\right]\left[\hat{\boldsymbol{H}}^2\right]...\left[\hat{\boldsymbol{H}}^m\right]\right] \tag{9}$$

or symbolically
$$\underset{(2N \text{ x } 2N)}{[\boldsymbol{S}]}\;\underset{(2N \text{ x } 2m)}{[\boldsymbol{H}]} = \underset{(2N \text{ x } 2m)}{\left[\hat{\boldsymbol{H}}\right]} \tag{10}$$

The system matrix $[\boldsymbol{S}]$ must satisfy the dynamic equation (5) and the harmonic relation (10). So $[\boldsymbol{S}]$ is solution of

$$[\boldsymbol{S}]\left[[\boldsymbol{H}]\,[\boldsymbol{X}]\right] = \left[\left[\hat{\boldsymbol{H}}\right][\boldsymbol{X}]\right] \tag{11}$$

Our objective is to solve the equation (11) exactly for the harmonic part and therefore we partition the matrices in equation (11) as

$$\begin{bmatrix} \underset{(2\text{m} \times 2\text{m})}{[\boldsymbol{S}_1]} & \underset{(2\text{m} \times 2\text{N-}2\text{m})}{[\boldsymbol{S}_2]} \\ \underset{(2\text{N-}2\text{m} \times 2\text{m})}{[\boldsymbol{S}_3]} & \underset{(2\text{N-}2\text{m} \times 2\text{N-}2\text{m})}{[\boldsymbol{S}_4]} \end{bmatrix} \begin{bmatrix} \underset{(2\text{m} \times 2\text{m})}{[\boldsymbol{H}_1]} & \underset{(2\text{m} \times \text{Ln})}{[\boldsymbol{X}_1]} \\ \underset{(2\text{N-}2\text{m} \times 2\text{m})}{[\boldsymbol{H}_2]} & \underset{(2\text{n-}2\text{m} \times \text{Ln})}{[\boldsymbol{X}_2]} \end{bmatrix} = \begin{bmatrix} \underset{(2\text{m} \times 2\text{m})}{\left[\hat{\boldsymbol{H}}_1\right]} & \underset{(2\text{m} \times \text{Ln})}{\left[\hat{\boldsymbol{Y}}_1\right]} \\ \underset{(2\text{N-}2\text{m} \times 2\text{m})}{\left[\hat{\boldsymbol{H}}_2\right]} & \underset{(2\text{n-}2\text{m} \times \text{Ln})}{\left[\hat{\boldsymbol{Y}}_2\right]} \end{bmatrix} \tag{12}$$

Equation (12) is first solved for $[\boldsymbol{S}_1]$ and $[\boldsymbol{S}_3]$ so that the harmonic part (10) is satisfied exactly, namely

$$\boldsymbol{S}_1 = \left(\hat{\boldsymbol{H}}_1 - \boldsymbol{S}_2\boldsymbol{H}_2\right)\boldsymbol{H}_1^{-1} \quad \text{and} \quad \boldsymbol{S}_3 = \left(\hat{\boldsymbol{H}}_2 - \boldsymbol{S}_4\boldsymbol{H}_2\right)\boldsymbol{H}_1^{-1} \tag{13}$$

The remainder of $[\boldsymbol{S}]$, namely $[\boldsymbol{S}_2]$ and $[\boldsymbol{S}_4]$, is then computed from the actual response signal as in the standard OMA procedure by solving in a least square sense

$$\begin{aligned} \boldsymbol{S}_2\left(\boldsymbol{H}_2\boldsymbol{H}_1^{-1}\boldsymbol{X}_1 - \boldsymbol{X}_2\right) &= \left(\hat{\boldsymbol{H}}_1\boldsymbol{H}_1^{-1}\boldsymbol{X}_1 - \hat{\boldsymbol{X}}_1\right) \\ \boldsymbol{S}_4\left(\boldsymbol{H}_2\boldsymbol{H}_1^{-1}\boldsymbol{X}_1 - \boldsymbol{X}_2\right) &= \left(\hat{\boldsymbol{H}}_2\boldsymbol{H}_1^{-1}\boldsymbol{X}_1 - \hat{\boldsymbol{X}}_2\right) \end{aligned} \tag{14}$$

Once $[\boldsymbol{S}]$ is known, the system eigenvalues are obtained by solving the eigenvalue problem. Note that $s = \pm i\omega_r$, $r = 1, 2..m$ will obviously be the solutions by construction.

4 Experiment

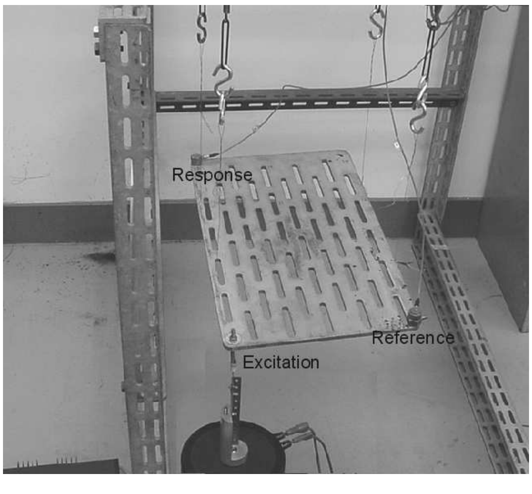

Figure 1: Experimental setup

The method above discussed was applied to the analysis of a rectangular steel plate in the laboratory. The plate was hanged using soft strings as shown in the Fig. 1. Input force was applied through a shaker to provide stationary noise excitation as well as harmonic loads at one of the corner of the free-hanging plate. The response acceleration was measured at the opposite corners of the plate. A SigLab 4-channel data acquisition system has been used for the experiment. Sampling of the signals were done at $2560Hz$ which is good to compute frequencies up to $1000Hz$. As discussed in the theory the correlation function, which is considered as the impulse response function has been computed by taking 1^{st} accelerometer signal as the reference signal and 2^{nd}accelerometer signal as the response signal. At first the plate was excited with white-noise and actual modal parameters (natural frequencies and associated damping) were computed. In all the examples, we have taken L to be 250 to maintain consistency of the results for comparison.

Mode	Frequency (Hz)	Damping (%)
1	141.679	0.341
2	153.277	0.208
3	270.400	0.335
4	328.089	0.210
5	408.440	0.139
6	436.498	0.159
7	560.528	0.477
8	623.449	0.463
9	678.042	0.331
10	731.776	0.544
11	814.525	0.927
12	844.645	0.318
13	892.599	0.294

Table 1: Frequencies and associated dampings (white noise only)

In Table 1 all the identified natural frequencies and associated damping have been listed within the frequency range of consideration. We identified one natural frequency at $270.40Hz$, whose associated damping is 0.335%. In the next section harmonic excitations will be applied in addition to the white noise and both standard SSTD and the modified SSTD method will be applied to extract modal parameters and they will be compared.

4.1 Harmonics at $274Hz$ and $276Hz$

In this experiment, two harmonic frequencies in addition to the white noise are present at $274Hz$ and $276Hz$, which are close to the $270.40Hz$ modal frequency of the system. In Fig. 2, two peaks at $274Hz$ and $276Hz$ are clearly observed from the power spectral density (PSD) plot. One secondary peak is present slightly behind $274Hz$ which might be representing the $270.40Hz$ natural frequencies of the system. As described in the theory, we introduced four additional rows of sin and cos terms to explicitly account for the two harmonic frequencies in the response matrices (7). Figure 2 shows the stability diagram superimposed on the auto-spectrum diagram. Auto-spectrum is applies to the signal from the reference (1^{st}) accelerometer signal. Modal parameters were computed with the standard SSTD and the modified SSTD for harmonics as presented in this paper.

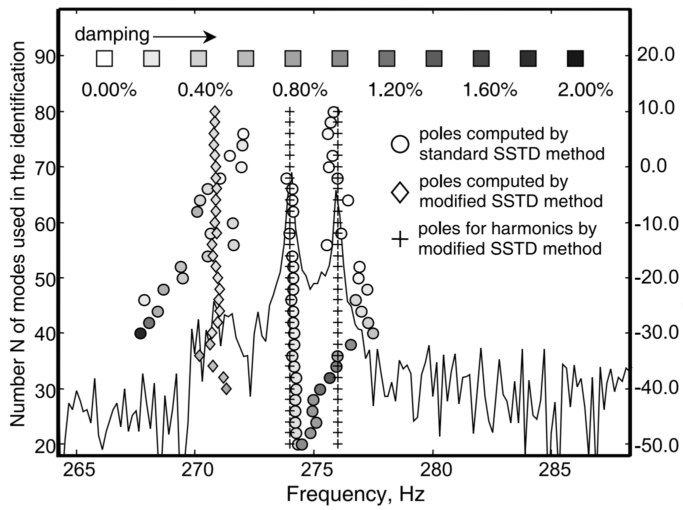

Figure 2: PSD of the acceleration for random loads and three additional harmonic excitations, and stabilization diagram for the standard SSTD and for the modified SSTD method to explicitly include harmonic components.

In the Fig. 2 different symbols have been used for the poles computed by standard SSTD and modified SSTD method. Since the presence of harmonics only creates problems in computing the modes which are close to the harmonic frequencies, Fig. 2 shows a zoomed portion of the full stability diagram. From Fig. 2 it can be seen that, while using the standard SSTD, the identified frequencies are associated to the two fundamental harmonic frequencies. Poles computed by the standard SSTD method have been represented by $\bigcirc$ signs. Note that the identified damping corresponding to the harmonic frequencies are not small, while there should be null in theory. Hence, in practice, it would be difficult at this point to assimilate the identified modes to harmonic responses and the analyst would probably conclude that the identified parameters correspond to true eigenmodes. At the location of natural frequency, i.e. $270.40Hz$ there are some poles which are computed by the standard SSTD method. But it can again be seen in the Fig. 2 that those poles are not stable as the modal order of the system changes as. Associated dampings are also not stable which can be observed from the color-depth.

While applying the modified SSTD method, we have introduced in the Hankel matrix two harmonic components of frequencies corresponding to the periodic excitations, namely $274Hz$ and $276Hz$. In the stabilization diagram shown in Fig. 2, $+$ signs are for the harmonic components of the computed poles. By construction of the Hankel matrix and computing procedure, harmonic frequencies are exactly matched and associated dampings are zero. It can be seen that there is only one more stability line with $\Diamond$ signs, which is in the region of $270.40Hz$, a natural frequency of the system. The consistency of the stability plot gives an indication of a pole location at that frequency. The modified SSTD method computes the pole as $270.711Hz$ and 0.30% damping, which are values quite close to the actual value of the eigenparameters computed while the plate was excited by noise only.

4.2 Harmonics at $277Hz$, $282Hz$ and $287Hz$

Here three harmonic frequencies in addition to the white noise are present at $277Hz$, $282Hz$ and $287Hz$, which are again close to the $270.40Hz$ modal frequency of the system. In Fig. 3 three peaks at $277Hz$, $282Hz$ and $287Hz$ are clearly observed from the power spectral density (PSD) plot. Again a secondary peak is present slightly before $277Hz$ which might be indicating the

$270.40Hz$ natural frequencies of the system. As described in the theory, six additional rows of sin and cos terms have been introduced to explicitly account for the two harmonic frequencies in the response matrices (7). Figure 3 shows the stability diagram.

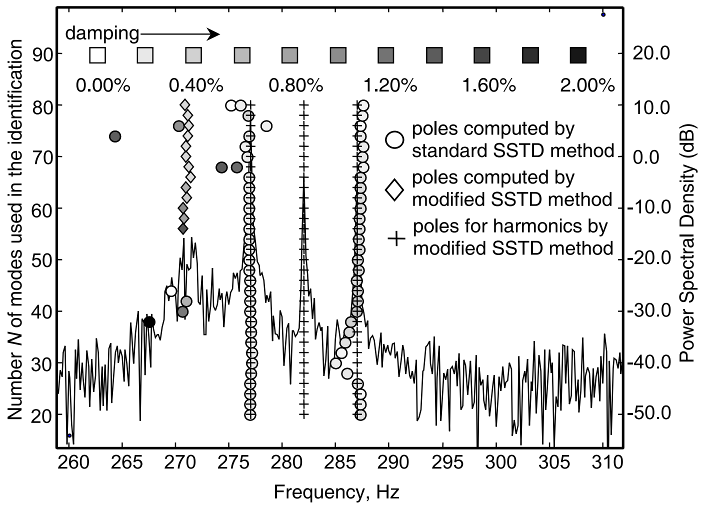

Figure 3: PSD of the acceleration for random loads and three additional harmonic excitations, and stabilization diagram for the standard SSTD and for the modified SSTD method to explicitly include harmonic components.

Figure 3 is a zoomed portion of the full stability diagram. From Fig. 3 it can be seen that, while using the standard SSTD, the identified frequencies are associated to the two harmonic frequencies at $277Hz$ and $287Hz$. There is even no identified pole for the $282Hz$ harmonic frequency. Again dampings computed by the standard SSTD method at the harmonic frequencies are quite significant, while they should be null according to theory. Except stability lines at the two harmonic excitation frequency locations, no other stability line is computed by the standard SSTD method. Hence it can be concluded that standard SSTD method completely fails to identify any pole location in the frequency range of interest.

While applying the modified SSTD method, we have introduced in the Hankel matrix three harmonic components of frequencies corresponding to the periodic excitations, $277Hz$, $282Hz$ and $287Hz$. Like before in the stabilization diagram in Fig. 3, + signs are for the harmonic components of the computed poles and by construction harmonic frequencies are exactly matched and associated damping is zero. It can be seen that there is again only one more stability line with $\Diamond$ signs, which is in the region of the $270.40Hz$ natural frequency of the system. The consistency of the stability plot gives an indication of a pole at that location. The modified SSTD method computes the pole as $270.320Hz$ and 0.32% damping, which are values quite close to the actual values.

5 Conclusion

In this work, we have considered the SSTD algorithm to identify modal parameters from measurements for in-operation modal analysis. In particular, a modification is introduced in the identification procedure to explicitly account for harmonic components that might be present in addition to the response to stationary white noise. The proposed modified SSTD method allows to identify accurately eigenfrequencies of the system even when the harmonic frequencies are close to an eigenfrequency.

It is however very important that the harmonic frequencies be provided very accurately to the algorithm. In the case when the frequencies introduced in the algorithm are very different from the exact harmonic frequencies, the modified SSTD becomes equivalent to the standard SSTD method. So the modified SSTD method helps to compute modal parameters, which are influenced due to the harmonics. However it has no significant impact on the other modes, which are computed correctly both by the standard SSTD and modified SSTD methods.

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